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2006 J. Phys. A: Math. Gen. 39 3763

(http://iopscience.iop.org/0305-4470/39/14/018)

Institute of Physics Publishing Journal of Physics A: Mathematical and General
J. Phys. A: Math. Gen. **39** (2006) 3763–3769 doi:10.1088/0305-4470/39/14/018

Feynman problem in the noncommutative case

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Received 9 January 2006, in final form 6 February 2006
Published 22 March 2006
Online at stacks.iop.org/JPhysA/39/3763

Abstract
In the context of Feynman's derivation of electrodynamics, we show that noncommutativity allows particle dynamics other than the standard formalism of electrodynamics.

PACS numbers: 11.10.Nx, 45.20.Jj

1. Introduction

The Feynman procedure [1] to obtain Maxwell's equations in electrodynamics has been reviewed under different kind of settings, and several nontrivial and interesting generalizations are possible (see for instance [2–5, 7–9]). In general, the locality property that different coordinates commute is assumed. However, as pointed out by Jackiw [10], Heisenberg suggested in a letter to Peierls [11] that spatial coordinates may not commute; Peierls communicated the same idea to Pauli [12], who told it to Oppenheimer; eventually the idea arrived to Snyder [13] who wrote the first paper on the subject. On the other hand, the existence of a minimal length beyond which no strict localization is possible, the importance of the physics in noncommutative planes, the noncommutative Landau problem, Peierls substitution, and the fact that noncommutative field theory is relevant not only in string theory but also in condensed matters motivated a new interest on the subject during the last years.

Due to this increasing interest in noncommutative field theories, it is worthwhile to consider the noncommutative version of such procedure, where locality no longer holds, which has a better chance to find new kinds of particle dynamics, which after all, according to Dyson [1], was the original aim of Feynman. Such considerations were actually done in [14], but the argument given there seems to be inadequate or incomplete for two reasons: they only considered the case where the nonlocality is described by a coordinate-independent Moyal Bracket, whereas nowadays the non-constant (i.e. coordinate dependent) noncommutative spaces are gaining a lot of attention in the noncommutative realm, because of the appearance of such a type of noncommutativity in various contexts specially in string theory. Among the papers that invoke variable noncommutativity are [15–22]. On the other hand, the treatment in [14] is somewhat sloppy and the main conclusions are not correct, as we shall point out later on.

To avoid unnecessary complications due to operator ordering, we shall only discuss the classical analogue of the Feynman procedure in the noncommutative case. Accordingly, the appropriate setting would be in terms of Poisson brackets, which is regarded as the classical limit of the commutator of quantum observables. But we shall explore the different possibilities arising from the dependence of the fundamental brackets on the different sets of variables involved.

Let $\mathcal{F}(M)$ be the algebra of functions (the algebra of classical observables) on a manifold M (the classical state space). A *Poisson structure* on M is a real skew symmetric bilinear map $\{\cdot,\cdot\}\ \mathcal{F}(M)\times\mathcal{F}(M)\to\mathcal{F}(M)$ satisfying the Jacobi identity:

$$\{F,\{G,H\}\}+\{H,\{F,G\}\}+\{G,\{H,F\}\}=0, \qquad \forall F,G,H\in\mathcal{F}(M),$$

and such that the map $X_F=\{\cdot,F\}$ is a derivation of the Lie algebra $\mathcal{F}(M)$, for each $F\in\mathcal{F}(M)$, in other words, X_F is a vector field, usually called a *Hamiltonian vector field*, and F is said to be the *Hamiltonian* of X_F. This second property, called the Leibnitz' rule, is important as there are many examples of Lie algebra structures on $\mathcal{F}(M)$ that do not satisfy Leibnitz' rule.

In particular, if ξ^a denotes a set of local coordinates on M, then, using the summation index convention, the local coordinate expression of the Poisson bracket becomes

$$\{F,G\}=\{\xi^a,\xi^b\}\frac{\partial G}{\partial\xi^b}\frac{\partial F}{\partial\xi^a}. \tag{1}$$

2. The velocity-independent case

In this section we study the Feynman argument in the framework of a tangent bundle, in the case where the bracket is nonlocal; in other words, we do not suppose that the variables on the configuration space commute. So we assume that the Poisson manifold M is the tangent bundle TQ of an n-dimensional configuration space Q, with local coordinates $x^i,\dot{x}^i$, for $i=1,\ldots,n=\dim Q$. Thus a general Poisson bracket on TQ is locally given by

$$\{F,G\}=\{x^i,x^j\}\frac{\partial G}{\partial x^j}\frac{\partial F}{\partial x^i}+\{x^i,\dot{x}^j\}\frac{\partial G}{\partial\dot{x}^j}\frac{\partial F}{\partial x^i}+\{\dot{x}^i,x^j\}\frac{\partial G}{\partial x^j}\frac{\partial F}{\partial\dot{x}^i}+\{\dot{x}^i,\dot{x}^j\}\frac{\partial G}{\partial\dot{x}^j}\frac{\partial F}{\partial\dot{x}^i}. \tag{2}$$

To shorten the length of the computations and simplify the mathematics of the problem we shall consider only autonomous systems, so the fields below do not depend directly on the time t, but our arguments can be extended to non-autonomous systems and more general contexts. We first consider a bracket such that

$$\{x^i,x^j\}=g_{ij}(x), \tag{3}$$

where g_{ij} is an arbitrary skewsymmetric matrix of functions, fulfilling the constraints that a Poisson bracket satisfying the Leibniz rule impose. In other words, we examine the possibility of a bracket without the locality property; a condition needed, for instance, in a classical description of a massless particle [23]. We also require

$$m\{x^i,\dot{x}^j\}=\delta_{ij}, \tag{4}$$

so this part of the Poisson bracket is the same as in the commutative case, considered by Feynman. Now, the Jacobi identity

$$\{x^i,\{x^j,\dot{x}^k\}\}+\{\dot{x}^k,\{x^i,x^j\}\}+\{x^j,\{\dot{x}^k,x^i\}\}=0$$

entails, upon using (4), and $\partial g_{ij}/\partial\dot{x}^k=0$,

$$0=\{\dot{x}^k,g_{ij}\}=\{\dot{x}^k,x^l\}\frac{\partial g_{ij}}{\partial x^l}+\{\dot{x}^k,\dot{x}^l\}\frac{\partial g_{ij}}{\partial\dot{x}^l}=-\frac{1}{m}\frac{\partial g_{ij}}{\partial x^k}. \tag{5}$$

Thus, the matrix g_{ij} is a constant skewsymmetric 3×3 matrix, and nonconstant matrices will only be possible if one assume dependence of g on the dotted variables, but we explore this possibility in the next section.

To continue with Feynman's argument we further assume Newton's equations:

$$m\ddot{x}^j = F^j(x, \dot{x}). \tag{6}$$

In other words, we assume that the equations of motion can be written as

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \{x^i, H\} = \dot{x}^i, \tag{7}$$

$$\frac{\mathrm{d}\dot{x}^i}{\mathrm{d}t} = \{\dot{x}^i, H\} = \frac{1}{m}F^i(x, \dot{x}), \tag{8}$$

with $\{\cdot, \cdot\}$ being a Poisson bracket and H a Hamiltonian function both to be determined. Note however that as we assumed the nonlocality property of the Poisson bivector, such bivector cannot be associated with a symplectic structure defined by a regular Lagrangian, because the locality assumption is equivalent to the vanishing of the symplectic form ω_L on a pair of vertical fields, which is a necessary condition for the existence of a regular Lagrangian [24, 25].

Now, we restrict ourselves to the three-dimensional case dim $Q = 3$, and take the time derivative of (4) to obtain

$$0 = m\{\dot{x}^i, \dot{x}^j\} + \{x^i, \ddot{x}^j\} = m\{\dot{x}^i, \dot{x}^j\} + \{x^i, F^j\},$$

therefore

$$\{x^i, F^j\} = -m\{\dot{x}^i, \dot{x}^j\} = m\{\dot{x}^j, \dot{x}^i\} = -\{x^j, F^i\},$$

so $\{x^i, F^j\}$ is skewsymmetric and we can define a field $B_k(x, \dot{x})$ that, in analogy with the commutative case, we may call the magnetic field, by means of

$$-\frac{1}{m}\{x^i, F^j\} = \{\dot{x}^i, \dot{x}^j\} = \frac{1}{m^2}\varepsilon_{ijk}B_k(x, \dot{x}), \tag{9}$$

where ε_{ijk} denotes the fully skewsymmetric Levi-Civita tensor, for which $\varepsilon_{123} = 1$; so, for instance,

$$B_3 = m^2\{\dot{x}^1, \dot{x}^2\}. \tag{10}$$

Now, the Jacobi identities with one position and two velocities entail

$$\{x^i, B_j\} = 0,$$

and the local expression (2) gives

$$0 = \{x^i, B_j\} = g_{ik}\frac{\partial B_j}{\partial x^k} + \frac{1}{m}\frac{\partial B_j}{\partial \dot{x}^i}. \tag{11}$$

In the commutative case, i.e. when $g_{ik} \equiv 0$, (11) implies that B_j is independent of the $\dot{x}$'s, but in our setting this is not necessarily true. However, notice that, for instance

$$\{\dot{x}^3, B_3\} = m^2\{\dot{x}^3, \{\dot{x}^1, \dot{x}^2\}\}.$$

Thus, the Jacobi identity with three different velocities gives

$$\{\dot{x}^i, B_i\} = 0. \tag{12}$$

Once again the local expression of the Poisson bracket gives

$$m\{\dot{x}^i, B_j\} = -\frac{\partial B_j}{\partial x^i} + m\{\dot{x}^i, \dot{x}^k\}\frac{\partial B_j}{\partial \dot{x}^k} = -\frac{\partial B_j}{\partial x^i} + \frac{1}{m}\varepsilon_{ilk}B_k\frac{\partial B_j}{\partial \dot{x}^l},$$

and then we can rewrite (12) as

$$\operatorname{div} \mathbf{B} = -\frac{1}{m}\mathbf{B} \cdot \dot{\nabla} \times \mathbf{B}, \tag{13}$$

upon using the notation $\dot{\nabla} = (\partial/\partial\dot{x}^1, \partial/\partial\dot{x}^2, \partial/\partial\dot{x}^3)$. This is the equation that replaces the Maxwell equation $\operatorname{div} \mathbf{B} = 0$, describing the absence of monopoles in the noncommutative case. In the particular case when the field $\mathbf{B}$ is independent of the $\dot{x}$'s, the previous equation (13) reduces indeed to the Maxwell equation

$$\operatorname{div} \mathbf{B} = 0.$$

Now, we mentioned already that $\mathbf{B}$ may very well depend on the variables $\dot{x}$, but even if we assume that the field $\mathbf{B}$ is independent of the $\dot{x}$'s, from (11) we see that $\mathbf{B}$ can still depend on the variables x, since the matrix g_{ij}, being a constant skewsymmetric 3×3 matrix, is singular. Therefore, the conclusion in [14] that the conditions (3), (4) and (6) entail static Maxwell equations is wrong. One of the problems in [14] is that in the noncommutative space that they are using, which is neither explicitly defined nor described, it is not clear at all the meaning of the dotted variables.

On the other hand, in the quest of an equation similar to the second Maxwell equation, we define another field $\mathbf{E}$, the electric field, by $E^j = F^j - \varepsilon_{jkl}\dot{x}^k B_l$. This makes sense in the commutative case because, there, $\mathbf{B}$ is certainly independent of the $\dot{x}$'s and, as we shall see in a moment, (9) implies that $\mathbf{F}$ is at most linear in the $\dot{x}$'s variables, but again this is not necessarily what happens in our setting, even if we assume independence of $\mathbf{B}$ on the $\dot{x}$'s variables. Indeed, from (9) and (11) we obtain

$$\begin{aligned}
\{x^i, E_j\} &= \{x^i, F^j - \varepsilon_{jkl}\dot{x}^k B_l\} \\
&= \{x^i, F^j\} - \varepsilon_{jkl}\{x^i, \dot{x}^k\}B_l - \varepsilon_{jkl}\dot{x}^k\{x^i, B_l\} \\
&= \{x^i, F^j\} - \frac{1}{m}\varepsilon_{jkl}B_l = 0;
\end{aligned}$$

therefore, as claimed, in the commutative case the field $\mathbf{E}$, so defined, is independent of the velocities.

Following the commutative case, we take the derivative with respect to t of (10):

$$\begin{aligned}
\dot{x}^l\frac{\partial B_k}{\partial x^l} + \frac{1}{m}F^l\frac{\partial B_k}{\partial \dot{x}^l} &= \frac{m^2}{2}\varepsilon_{ijk}(\{\dot{x}^i, F^j\} + \{F^i, \dot{x}^j\}) = m\varepsilon_{ijk}\{F^i, \dot{x}^j\} \\
&= m\varepsilon_{ijk}(\{E^i, \dot{x}^j\} + \varepsilon_{iln}\{\dot{x}^l, \dot{x}^j\}B_n + \varepsilon_{iln}\dot{x}^l\{B_n, \dot{x}^j\}).
\end{aligned} \tag{14}$$

Now, the local expressions of the brackets give

$$\begin{aligned}
m\varepsilon_{ijk}\{E^i, \dot{x}^j\} &= m\varepsilon_{ijk}\left(\{x^l, \dot{x}^j\}\frac{\partial E^i}{\partial x^l} + \{\dot{x}^l, \dot{x}^j\}\frac{\partial E^i}{\partial \dot{x}^l}\right) \\
&= \varepsilon_{ijk}\left(\frac{\partial E^i}{\partial x^j} + \frac{1}{m}\varepsilon_{ljn}B_n\frac{\partial E^i}{\partial \dot{x}^l}\right) \\
&= \varepsilon_{ijk}\frac{\partial E^i}{\partial x^j} + \frac{1}{m}(\delta_{il}\delta_{kn} - \delta_{in}\delta_{kl})B_n\frac{\partial E^i}{\partial \dot{x}^l} \\
&= \varepsilon_{ijk}\frac{\partial E^i}{\partial x^j} + \frac{1}{m}\left(B_k\frac{\partial E^l}{\partial \dot{x}^l} - B_n\frac{\partial E^n}{\partial \dot{x}^k}\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
m\varepsilon_{ijk}\varepsilon_{iln}\{\dot{x}^l, \dot{x}^j\}B_n &= m(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl})\{\dot{x}^l, \dot{x}^j\}B_n \\
&= -m\{\dot{x}^k, \dot{x}^j\}B_j \\
&= -\frac{1}{m}\varepsilon_{kjl}B_l B_j = 0,
\end{aligned}$$

on account of (9). Also, using (12), we have

$$\begin{aligned}
m\varepsilon_{ijk}\varepsilon_{iln}\dot{x}^n\{B_l,\dot{x}^j\} &= m(\delta_{jn}\delta_{kl}-\delta_{jl}\delta_{kn})\dot{x}^n\{B_l,\dot{x}^j\}\\
&= m(\dot{x}^n\{B_k,\dot{x}^n\}-\dot{x}^k\{B_l,\dot{x}^l\})\\
&= m\dot{x}^n\{B_k,\dot{x}^n\}\\
&= m\left(\dot{x}^n\{x^l,\dot{x}^n\}\frac{\partial B_k}{\partial x^l}+\dot{x}^n\{\dot{x}^l,\dot{x}^n\}\frac{\partial B_k}{\partial \dot{x}^l}\right)\\
&= \dot{x}^l\frac{\partial B_k}{\partial x^l}+\frac{1}{m}\dot{x}^n\varepsilon_{lnr}B_r\frac{\partial B_k}{\partial \dot{x}^l}\\
&= \dot{x}^l\frac{\partial B_k}{\partial x^l}+\frac{1}{m}F^l\frac{\partial B_k}{\partial \dot{x}^l}-\frac{1}{m}E^l\frac{\partial B_k}{\partial \dot{x}^l}.
\end{aligned}$$

Collecting all together, we see that (14) reduces to

$$\varepsilon_{ijk}\frac{\partial E^i}{\partial x^j}=\frac{1}{m}\left(E^l\frac{\partial B_k}{\partial \dot{x}^l}+B_l\frac{\partial E^l}{\partial \dot{x}^k}-B_k\frac{\partial E^l}{\partial \dot{x}^l}\right). \tag{15}$$

In vector form this can be rewritten as

$$(\mathrm{rot}\,\mathbf{E})_k+\frac{1}{m}\left((\mathbf{E}\cdot\dot{\nabla})B_k+\mathbf{B}\cdot\frac{\partial\mathbf{E}}{\partial\dot{x}^k}-(\dot{\nabla}\cdot\mathbf{E})B_k\right)=0, \tag{16}$$

which is what replaces the Maxwell equation corresponding to Faraday's law, in the setting suggested at the beginning of this section. Had we assumed that $\mathbf{F}$ in (6) can depend on t, and therefore $\mathbf{B}$ defined in (9) also is t dependent, then the term $\partial B_k/\partial t$ would appear in the left-hand side of (14) and in the right-hand side of (15). Of course, as explicitly remarked in Dyson's paper [1], only the homogeneous Maxwell equations can be obtained.

Finally, we point out that had we assumed that the fields $\mathbf{B}$ and $\mathbf{E}$ do not depend on the $\dot{x}$'s (so $\mathbf{F}$ is actually a Lorentz force), then (13) and (16) would exactly be the usual Maxwell equations, so in the limit we have a smooth transition into the commutative case, contrary to what is claimed in [14]. However, here the Lorentz force condition would be an extra assumption, not a consequence as in the commutative case considered in [1].

3. Velocity-dependent Poisson brackets

We now return to the case where the matrix $g_{ij}=g_{ij}(x,\dot{x})$ in (3) also depends on the variables $\dot{x}$. Then g_{ij} no longer need to be a constant matrix, as now (5) rather imposes on g_{ij} the condition

$$0=-\frac{1}{m}\frac{\partial g_{ij}}{\partial x^k}+\{\dot{x}^k,\dot{x}^l\}\frac{\partial g_{ij}}{\partial \dot{x}^l}.$$

Moreover, the Jacobi identity on x^i, x^j and x^k reduces to

$$\begin{aligned}
0&=\{x^i,g_{jk}\}+\{x^k,g_{ij}\}+\{x^j,g_{ki}\}\\
&=g_{il}\frac{\partial g_{jk}}{\partial x^l}+g_{kl}\frac{\partial g_{ij}}{\partial x^l}+g_{jl}\frac{\partial g_{ki}}{\partial x^l}+\frac{1}{m}\left(\frac{\partial g_{jk}}{\partial \dot{x}^i}+\frac{\partial g_{ij}}{\partial \dot{x}^k}+\frac{\partial g_{ki}}{\partial \dot{x}^j}\right),
\end{aligned}$$

which gives exactly one more constraint on the g_{ij}'s, since the skewsymmetry property of g_{ij} entails that a permutation of the indices gives the same equation as for $i=1$, $j=2$ and $k=3$ when the permutation is even, and negative the expression if the permutation is odd. Furthermore, in the previous section we did not use the fact that the g's were constant; therefore by the same token we obtain also for $g_{ij}(x,\dot{x})$ the generalized Maxwell equations (13) and (16).

On the other hand, even though condition (4) simplified matters quite a bit, it may be useful, in some settings, to modify also this condition. Thus we now address the problem when

$$\{x^i, x^j\} = g_{ij}(x, \dot{x}), \tag{17}$$

and

$$m\{x^i, \dot{x}^j\} = \delta_{ij} + f_{ij}(x, \dot{x}), \tag{18}$$

where f_{ij} is another matrix compatible with the Poisson bracket properties, which now impose several relations among the g_{ij}'s and the f_{ij}'s, again the g_{ij}'s need not be constants.

In principle, there is no need to impose a special condition on f_{ij}, but the parallelism with the computation of the previous section is more transparent if one assumes, as we do, that f_{ij} is skewsymmetric. In [14] a particular instance of this situation was considered, but they assumed that the variables $\dot{x}$ are functions of the x^i's, a hypothesis without much physical justification, they assume a special form of the f_{ij}'s which is completely unnecessary, and they place their argument in the constant noncommutative case.

Once more, taking the derivative with respect to t we obtain

$$\frac{\mathrm{d}f_{ij}}{\mathrm{d}t} = m\{\dot{x}^i, \dot{x}^j\} + \{x^i, F^j\},$$

therefore

$$\{\dot{x}^i, \dot{x}^j\} = \frac{1}{m}\left(\frac{\mathrm{d}f_{ij}}{\mathrm{d}t} - \{x^i, F^j\}\right),$$

and since f_{ij} is skewsymmetric,

$$\{x^i, F^j\} = -\{x^j, F^i\},$$

so a field $\mathbf{B}$ can be defined as in (9), and exactly the same computations can be performed, leading to some equations a bit more involved, but similar to (13) and (16). We see no point in repeating the calculations.

In this context the equations of motion become

$$\dot{x}^i = \{x^i, x^j\}\frac{\partial H}{\partial x^j} + \{x^i, \dot{x}^j\}\frac{\partial H}{\partial \dot{x}^j}, \qquad F^i = \{\dot{x}^i, x^j\}\frac{\partial H}{\partial x^j} + \{\dot{x}^i, \dot{x}^j\}\frac{\partial H}{\partial \dot{x}^j},$$

which are more complicated than the classical ones, but, in principle, a Hamiltonian description is still possible in the noncommutative setting.

We conclude that noncommutativity does allow other dynamics than the standard formalism of electrodynamics.

Acknowledgments

We thank José Gracia-Bondía and Giuseppe Marmo for useful conversations. JFC acknowledges financial support from research projects BFM2003-02532 and DGA-GRUPOS CONSOLIDADOS E24/1. HF thanks the Departamento de Física Teórica de la Universidad de Zaragoza for its hospitality and acknowledges support from the Vicerectoría de Investigación of the Universidad de Costa Rica.

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